

# Description of Generalized Albanese Varieties by Curves

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## Abstract

Let  $X$  be a projective variety over an algebraically closed base field, possibly singular. The aim of this paper is to show that the generalized Albanese variety  $\text{Alb}(X, X_{\text{sing}})$  of Esnault-Srinivas-Viehweg can be computed from one general curve  $C$  in  $X$ , if the base field is of characteristic 0. We illustrate this by an example, which we also use to unravel some mysterious properties of  $\text{Alb}(X, X_{\text{sing}})$ .

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## 0 Introduction

In [Ru1] the author considered generalized Albanese varieties  $\text{Alb}_{\mathcal{F}}(X)$  associated with categories of rational maps from a variety  $X$  to algebraic groups. If such a generalized Albanese variety is generated by a curve  $C$ , then the dimension of  $\text{Alb}_{\mathcal{F}}(X)$  is bounded by the dimension of the generalized Albanese of the curve  $C$ , which is easy to compute. For example if  $\text{Alb}_{\mathcal{F}}(X)$  is the classical Albanese  $\text{Alb}(X)$  of a smooth proper variety  $X$  or the Albanese of Esnault-Srinivas-Viehweg  $\text{Alb}(X, X_{\text{sing}})$  of a (singular) projective variety  $X$  (see [ESV], cf. [Ru1]), then  $\text{Alb}(C)$  resp.  $\text{Alb}(C, C_{\text{sing}})$  is isomorphic to the Picard variety  $\text{Pic}^0 C$  of  $C$ . In this way the existence of the classical Albanese was shown in [Lg] and the existence of the Albanese of Esnault-Srinivas-Viehweg in [ESV].

The purpose of the present paper is to show that the functorial description of generalized Albanese varieties from [Ru1] is not only a purely theoretical one, but allows a concrete computation, using the interplay with curves. Here we restrict ourselves to the case that the base field  $k$  is algebraically closed of characteristic 0. The Albanese of Esnault-Srinivas-Viehweg  $\text{Alb}(X, X_{\text{sing}})$  is an extension of the classical Albanese  $\text{Alb}(\tilde{X})$ , where  $\tilde{X} \rightarrow X$  is a resolution of singularities, by an affine group  $L_X$ , whose Cartier dual  $\underline{\text{Div}}_{\tilde{X}/X}^0$  is described in [Ru1, Prop.s 3.23, 3.24]. The main result of this work is a significant simplification of the presentation of  $\underline{\text{Div}}_{\tilde{X}/X}^0$  (Theorem 3.8). Moreover, the functorial description allows to explain some pathological properties of the Albanese of Esnault-Srinivas-Viehweg. This is accomplished in an example (Section 4).

### 0.1 Leitfaden

**Section 1.** We recall some facts about the Picard variety of curves that allow us to compute  $\text{Pic}^0 C$  of a singular curve  $C$ . Here we decompose  $\text{Pic}^0 C$  as an extension of the Picard variety  $\text{Pic}^0 \tilde{C}$  of the normalization  $\tilde{C}$  of  $C$  by a linear group  $L$  that takes care of the singularities of  $C$ .

**Section 2.** We consider a formal group  $\mathcal{F} \subset \underline{\text{Div}}_X$  of relative Cartier divisors on  $X$  and a curve  $C$  in  $X$ . We give a sufficient condition for the injectivity of the restriction map  $\mathcal{F} \rightarrow \underline{\text{Div}}_C$  from  $\mathcal{F} \subset \underline{\text{Div}}_X$  into the group sheaf of relative Cartier divisors on  $C$  (Lemma 2.2).

**Section 3.** The Albanese of Esnault-Srinivas-Viehweg  $\text{Alb}(X, X_{\text{sing}})$  is the dual of the 1-motivic functor  $[\underline{\text{Div}}_{\tilde{X}/X}^0 \rightarrow \text{Pic}_{\tilde{X}}^0]$ , where  $\tilde{X} \rightarrow X$  is a resolution of singularities, see [Ru1, Theorem 0.1]. Here  $\underline{\text{Div}}_{\tilde{X}/X}^0$  is the “kernel of the push-forward of relative Divisors from  $\tilde{X}$  to  $X$ ”, if  $X$  is a curve (see [Ru1, Proposition 3.23]). For higher dimensional  $X$  the definition of  $\underline{\text{Div}}_{\tilde{X}/X}^0$  is derived from the one for curves by intersecting the formal groups  $\underline{\text{Div}}_{\tilde{C}/C}^0$  associated with curves  $C$  in  $X$ , where the intersection ranges over all Cartier curves  $C$  in  $X$  relative to the singular locus of  $X$  (see [Ru1, Proposition 3.24]). So a priori this object looks hard to grasp. We explain how  $\underline{\text{Div}}_{\tilde{X}/X}^0$  can be computed from one single complete intersection curve in  $X$  (Corollary 3.7). Moreover, the curves with this property are dense in any space of sufficiently ample complete intersection curves (Theorem 3.8).

Sections 1, 2 and 3 provide the necessary tools in order to reduce the computation of  $\text{Alb}(X, X_{\text{sing}})$  to the curve case. This is demonstrated in an example

**Section 4.** The classical Albanese of smooth projective varieties  $X_i$  is compatible with products, i.e.  $\text{Alb}(\prod X_i) = \prod \text{Alb}(X_i)$ . More generally, all universal objects for categories of rational maps to *semi-abelian varieties* have this property. However, due to additive subgroups it is possible for singular projective varieties  $X_i$  that  $\dim \text{Alb}(\prod X_i, (\prod X_i)_{\text{sing}}) > \sum \dim \text{Alb}(X_i, (X_i)_{\text{sing}})$ . Moreover, if  $X$  is a smooth projective variety of dimension  $d$  and  $\mathcal{L}$  a very ample line bundle on  $X$ , then for a complete intersection  $C$  of  $d-1$  general divisors in the linear system  $|\mathcal{L}|$  the Gysin map  $\text{Alb}(C) \rightarrow \text{Alb}(X)$  will be surjective. This is not true in general for the Albanese of Esnault-Srinivas-Viehweg of a singular  $X$ , but a sufficiently high power  $\mathcal{L}^{\otimes N}$  will again have this property.

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Let  $C$  be a projective curve over a field  $k$ . The fact that divisors equal 0-cycles on  $C$  yields an identification of the Picard scheme  $\mathrm{Pic} C$  with the relative Chow group  $\mathrm{CH}_0(C, C_{\mathrm{sing}})$  from [LW].

Let  $C$  be a curve over a perfect field.

**Definition 1.1.** A point  $p$  of a curve  $C$  is called an *ordinary multiple point*, if it marks a transversal crossing of smooth formal local branches. More precisely,  $p \in C$  is an *ordinary  $m$ -ple point* if  $\hat{\mathcal{O}}_{C,p} \cong k[[t_1, \dots, t_m]] / \sum_{i \neq j} (t_i t_j)$ .

**Definition 1.2.** The normalization  $\nu : \tilde{C} \rightarrow C$  factors as  $\tilde{C} \xrightarrow{\sigma} C' \xrightarrow{\rho} C$  for a unique curve  $C'$  which is homeomorphic to  $C$  and has only ordinary multiple points as singularities, see [BLR, Section 9.2, p. 247].  $C'$  is called the *largest curve homeomorphic to  $C$* , or the *semi-normalization* of  $C$ .

Since  $C$  is reduced, the smooth locus is dense in  $C$ , hence the singular locus  $S$  is finite. The curve  $C'$  is obtained from  $\tilde{C}$  by identifying the points  $\tilde{p}_i \in \tilde{C}$  lying over  $p \in S$ . For an explicit description of  $C'$  see [BLR, Section 9.2, p. 247].

**Notation 1.3.** We write  $\mathcal{O} := \mathcal{O}_C$ ,  $\mathcal{O}' := \rho_* \mathcal{O}_{C'}$ ,  $\tilde{\mathcal{O}} := \nu_* \mathcal{O}_{\tilde{C}}$ .

## 1.2 Decomposition of the Picard Variety

**Proposition 1.4.** Let  $C'$  be a connected projective curve having only ordinary multiple points as singularities. Let  $\tilde{C}$  be the normalization of  $C'$ . Then  $\text{Pic}^0 C'$  is an extension of the abelian variety  $\text{Pic}^0 \tilde{C}$  by a torus  $T$ . If  $k$  is algebraically closed,  $T \cong (\mathbb{G}_m)^t$  is a split torus of rank  $t = \sum_{m \geq 1} (m-1) \#S_m - \# \text{Cp}(C') + 1$ , where  $\text{Cp}(C')$  is the set of irreducible components of  $C'$  and  $S_m$  is the set of  $m$ -ple points (see Definition 1.1).

**Proof.** (See also [BLR, Section 9.2, Proposition 10] for the first statement.) If  $S$  denotes the singular locus of  $C'$  and  $\text{Cp}(\tilde{C})$  the set of components of  $\tilde{C}$ , we obtain from  $1 \rightarrow \mathcal{O}'^* \rightarrow \tilde{\mathcal{O}}^* \rightarrow \mathcal{Q}^* \rightarrow 1$  with  $\mathcal{Q}^* = \prod_{p \in S} (\tilde{\mathcal{O}}_p)^* / (\mathcal{O}'_p)^*$  the long exact cohomology sequence

$$1 \rightarrow k_{C'}^* \rightarrow \prod_{Z \in \text{Cp}(\tilde{C})} k_Z^* \rightarrow \prod_{p \in S} T_p(k) \rightarrow \text{Pic}(C') \rightarrow \prod_{Z \in \text{Cp}(\tilde{C})} \text{Pic}(Z) \rightarrow 1$$

where  $k_{C'}^* = H^0(C', \mathcal{O}_{C'}^*)$ ,  $k_Z^* = H^0(Z, \mathcal{O}_Z^*)$ , and

$$T_p(k) = (\tilde{\mathcal{O}}_p)^* / (\mathcal{O}'_p)^* = \frac{\prod_{q \rightarrow p} k(q)^*}{k(p)^*} \cong (k^*)^{m_p-1}$$

since each  $p \in S$  is an ordinary multiple point (see Definition 1.1). Here  $\prod_{p \in S} T_p(k)$  maps to the connected component of the identity of  $\text{Pic}(C')$ . Then the affine part  $T$  of  $\text{Pic}^0 C'$  is the torus given by

$$T(k) = \text{coker} \left( \prod_{Z \in \text{Cp}(\tilde{C})} k_Z^* \rightarrow \prod_{p \in S} T_p(k) \right) = \frac{\prod_{p \in S} T_p(k)}{\prod_{Z \in \text{Cp}(\tilde{C})} k_Z^* / k_{C'}^*} \cong (k^*)^t$$

with  $t = \sum_{m \geq 1} (m-1) \#S_m - \# \text{Cp}(\tilde{C}) + 1$  and  $\# \text{Cp}(\tilde{C}) = \# \text{Cp}(C')$ . ■

**Proposition 1.5.** Let  $C$  be a projective curve, let  $C'$  be the largest homeomorphic curve between  $C$  and its normalization  $\tilde{C}$ . Then  $\text{Pic}^0 C$  is an extension of  $\text{Pic}^0 C'$  by a unipotent group  $U$ . If  $k$  is algebraically closed,  $U$  is characterized by  $U(k) = \prod_{p \in S} (1 + \mathfrak{m}_{C',p}) / (1 + \mathfrak{m}_{C,p})$ , if moreover  $\text{char}(k) = 0$ , the exponential map yields an isomorphism  $U(k) \cong \prod_{p \in S} \mathfrak{m}_{C',p} / \mathfrak{m}_{C,p}$ , where  $S = C_{\text{sing}}$  is the singular locus of  $C$ .

**Proof.** (See also [BLR, Section 9.2, Proposition 9] for the first statement.) The exact sequence  $1 \rightarrow \mathcal{O}^* \rightarrow \mathcal{O}'^* \rightarrow \mathcal{Q}^* \rightarrow 1$  with  $\mathcal{Q}^* = \prod_{p \in S} \mathcal{O}'_p^* / \mathcal{O}_p^*$  yields the exact cohomology sequence  $1 \rightarrow H^0(\mathcal{Q}^*) \rightarrow H^1(\mathcal{O}^*) \rightarrow H^1(\mathcal{O}'^*) \rightarrow 1$ , since

$\rho : C' \rightarrow C$  is a homeomorphism, thus  $H^0(\mathcal{O}'^*) \cong H^0(\mathcal{O}^*)$ . Moreover it holds  $H^0(\mathcal{Q}^*) = \prod_{p \in S} (\mathcal{O}'_p)^* / \mathcal{O}_p^* = \prod_{p \in S} (1 + \mathfrak{m}_{C',p}) / (1 + \mathfrak{m}_{C,p})$ . As  $U(k) := H^0(\mathcal{Q}^*)$  is a connected unipotent group, the image of  $U(k)$  is contained in the connected component of the identity of  $\text{Pic}(C) = H^1(\mathcal{O}^*)$ . ■

**Proposition 1.6.** *Let  $C$  be a projective curve and  $\tilde{C}$  its normalization. Then the Picard variety  $\text{Pic}^0 C$  is an extension of the abelian variety  $\text{Pic}^0 \tilde{C}$  by an affine algebraic group  $L = T \times_k U$ , which is the product of a torus  $T$  and a unipotent group  $U$ . If  $k$  is algebraically closed,  $T$  and  $U$  are characterized as in Propositions 1.4 and 1.5.*

**Proof.** Follows directly from Propositions 1.4 and 1.5, cf. [BLR, Section 9.2, Corollary 11]. ■

**Remark 1.7.** Let  $C$  be a projective curve. Then there is a canonical isomorphism  $\text{Pic}^0 C \cong \text{Alb}(C, C_{\text{sing}})$ , see [ESV, Introduction]. In particular, the affine part  $L$  of  $\text{Pic}^0 C$  is Cartier dual to  $\underline{\text{Div}}_{\tilde{C}/C}^0$ , by [Ru1, Thm. 0.1] for  $\text{char}(k) = 0$  and by [Ru3, Thm. 0.1] in general.

## 2 Restriction to Curves

Let  $X$  be a regular projective variety of dimension  $d$  over an algebraically closed field  $k$  of characteristic 0.

Let  $V$  be a subvariety of  $X$ . Let  $\underline{\text{Div}}_X$  be the functor of relative Cartier divisors as described in [Ru1, No. 2.1]. There is a canonical class map  $\underline{\text{Div}}_X \rightarrow \underline{\text{Pic}}_X$ , set  $\underline{\text{Div}}_X^0 := \underline{\text{Div}}_X \times_{\underline{\text{Pic}}_X} \underline{\text{Pic}}_X^0$ .

$\underline{\text{Dec}}_{X,V}$  is the subfunctor of  $\underline{\text{Div}}_X$  consisting of those families of Cartier divisors whose support intersects  $V$  properly, i.e.  $\text{Supp}(\mathcal{D})$  (see [Ru1, Def. 2.10]) does not contain any associated point of  $V$  for all  $\mathcal{D} \in \underline{\text{Dec}}_{X,V}(R)$ ,  $R$  a finite dimensional  $k$ -algebra (see [Ru1, Def. 2.11]).  $? \cdot V : \underline{\text{Dec}}_{X,V} \rightarrow \underline{\text{Div}}_V$  is the pull-back of relative Cartier divisors from  $X$  to  $V$  (see [Ru1, Def. 2.12]).

Let  $\delta \in \text{Lie}(\underline{\text{Div}}_X) = \Gamma(\mathcal{K}_X / \mathcal{O}_X)$  be a deformation of the zero divisor in  $X$ . Then  $\delta$  determines an effective divisor by the poles of its local sections. Hence for each generic point  $\eta$  of height 1 in  $X$ , with associated discrete valuation  $v_\eta$ , the expression  $v_\eta(\delta)$  is well defined and  $v_\eta(\delta) \leq 0$ . Thus we obtain a homomorphism  $v_\eta : \text{Lie}(\underline{\text{Div}}_X) \rightarrow \mathbb{Z}$ .

**Definition 2.1.** For an ample line bundle  $\mathcal{L}$  on  $X$  and an integer  $c$  with  $1 \leq c \leq \dim X$  write  $|\mathcal{L}|^c = \mathbb{P}(H^0(X, \mathcal{L})) \times \dots \times \mathbb{P}(H^0(X, \mathcal{L}))$  ( $c$  copies). Let  $H_1, \dots, H_c \in |\mathcal{L}|$  and  $V = \bigcap_{i=1}^c H_i$ . By abuse of notation we write  $V \in |\mathcal{L}|^c$  instead of  $(H_1, \dots, H_c) \in |\mathcal{L}|^c$ .

**Lemma 2.2.** *Let  $\mathcal{F}$  be a formal subgroup of  $\underline{\text{Div}}_X^0$  s.t.  $\mathcal{F} \cong \mathbb{Z}^t \times_k (\widehat{\mathbb{G}}_a)^v$  for  $t, v \in \mathbb{N}$ , where  $\widehat{\mathbb{G}}_a$  denotes the completion of  $\mathbb{G}_a$  at 0. Let  $S$  be the set of generic points of  $\text{Supp}(\mathcal{F})$  and  $S_{\text{inf}}$  the corresponding set for  $\text{Supp}(\mathcal{F}_{\text{inf}})$ . If  $\eta$  is a generic point of height 1 in  $X$ , denote by  $E_\eta$  the associated prime divisor. For an ample line bundle  $\mathcal{L}$  on  $X$  there is an open dense  $U \subset |\mathcal{L}|^{d-1}$  such that  $(? \cdot C)|_{\mathcal{F}} : \mathcal{F} \rightarrow \underline{\text{Div}}_C$  is injective for  $C \in U$  if*

$$\#(C \cap E_\eta) \geq \dim_k (\text{Lie } \mathcal{F})_\eta^{-v}$$

for all  $\eta \in S_{\text{inf}}$  and all  $-\nu \in v_\eta(\text{Lie } \mathcal{F})$ , where  $(\text{Lie } \mathcal{F})_\eta$  is the image of the localization  $\text{Lie } \mathcal{F} \subset \Gamma(\mathcal{K}_X/\mathcal{O}_X) \rightarrow (\mathcal{K}_X/\mathcal{O}_X)_\eta$ ,  $\delta \mapsto [\delta]_\eta$  at the height 1 point  $\eta$ , and  $(\text{Lie } \mathcal{F})_\eta^{-\nu} = ((\text{Lie } \mathcal{F})_\eta \cap \mathfrak{m}_{X,\eta}^{-\nu}/\mathcal{O}_{X,\eta})/((\text{Lie } \mathcal{F})_\eta \cap \mathfrak{m}_{X,\eta}^{-\nu+1}/\mathcal{O}_{X,\eta})$ . Here  $(\mathcal{K}_X/\mathcal{O}_X)_\eta = \bigcup_{\nu > 0} \mathfrak{m}_{X,\eta}^{-\nu}/\mathcal{O}_{X,\eta}$  and  $\mathfrak{m}_{X,\eta}^{-\nu} = \{f \in \mathcal{K}_{X,\eta} \mid v_\eta(f) \geq -\nu\}$ .

**Proof.** Let  $C = \bigcap_{i=1}^{d-1} H_i$  be a complete intersection curve in  $|\mathcal{L}|^{d-1}$ . As an ample divisor  $H_i$  intersects each closed subscheme of codimension 1 and  $H_i$  restricted to  $H_1 \cap \dots \cap H_{i-1}$  is again ample for all  $i = 2, \dots, d-1$ , it follows by induction that  $C \cap \text{Supp}(\mathcal{D}) \neq \emptyset$  for all  $0 \neq \mathcal{D} \in \mathcal{F}(R)$ ,  $R$  a finite dimensional  $k$ -algebra. If the intersection points of  $C$  and  $\text{Supp}(D)$  are in general position for each  $0 \neq D \in \mathcal{F}(k)$ , then  $(? \cdot C)|_{\mathcal{F}(k)} : \mathcal{F}(k) \rightarrow \underline{\text{Div}}_C(k)$  is injective.

For the infinitesimal part of  $\mathcal{F}$  consider the following diagram:

$$\begin{array}{ccc} \Gamma(\mathcal{K}_X/\mathcal{O}_X) \supset \text{Lie } \mathcal{F} & \xrightarrow{? \cdot C} & \Gamma(\mathcal{K}_C/\mathcal{O}_C) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{\eta \in S} (\mathcal{K}_X/\mathcal{O}_X)_\eta \supset \text{im}(\text{Lie } \mathcal{F}) & \longrightarrow & \bigoplus_{q \in C} (\mathcal{K}_C/\mathcal{O}_C)_q. \end{array}$$

For each  $\eta \in S_{\text{inf}}$  choose a local parameter  $t_\eta$  of  $\mathfrak{m}_{X,\eta}$ . Since  $C$  intersects  $E_\eta$  properly, and if  $C$  intersects  $E_\eta$  transversally in general points, we may assume that each  $q \in C \cap E_\eta$  is a regular closed point of  $C$ . Then the image  $t_q \in \mathcal{O}_C$  of  $t_\eta \in \mathcal{O}_X$  is a local parameter of  $\mathfrak{m}_{C,q}$  and  $v_q(\text{Lie } \mathcal{F} \cdot C) = v_\eta(\text{Lie } \mathcal{F})$ . Set  $-n_\eta = \min \{v_\eta(\text{Lie } \mathcal{F})\}$ . We may consider  $\text{Lie } \mathcal{F} \cdot C$  as a  $k$ -linear subspace of the  $k$ -vector space

$$\bigoplus_{\eta \in S_{\text{inf}}} \bigoplus_{q \in C \cap E_\eta} t_q^{-n_\eta} \mathcal{O}_{C,q} / \mathcal{O}_{C,q}.$$

Then the map

$$\begin{aligned} \text{im}(\text{Lie } \mathcal{F} \rightarrow (\mathcal{K}_X/\mathcal{O}_X)_\eta) &\longrightarrow \bigoplus_{q \in C \cap E_\eta} t_q^{-n_\eta} \mathcal{O}_{C,q} / \mathcal{O}_{C,q} \\ f t_\eta^{-\nu} &\longmapsto \sum_{q \in C \cap E_\eta} [f]_q t_q^{-\nu} \end{aligned}$$

is injective if  $\dim_k (\text{Lie } \mathcal{F})_\eta^{-\nu} \leq \#(C \cap E_\eta)$  for all  $-\nu \in v_\eta(\text{Lie } \mathcal{F})$  and the intersection points of  $C$  and  $E_\eta$  are in general position.  $(? \cdot C)|_{\mathcal{F}} : \text{Lie } \mathcal{F} \rightarrow \text{Lie}(\underline{\text{Div}}_C)$  is injective if these maps are injective for all  $\eta \in S_{\text{inf}}$ . Since  $\mathcal{F} \cong \mathbb{Z}^t \times_k (\widehat{\mathbb{G}}_a)^v$  by assumption,  $\mathcal{F}$  is already determined by  $\mathcal{F}(k)$  and  $\text{Lie } \mathcal{F}$ . ■

### 3 Computation of the Kernel of the Push-forward of Divisors

Let  $X$  be a projective variety of dimension  $d$  over an algebraically closed field  $k$  of characteristic 0. Let  $S$  denote the singular locus of  $X$ . For a line bundle  $\mathcal{L}$  on  $X$  we define  $|\mathcal{L}|_S = \{H \in |\mathcal{L}| \mid H \text{ intersects } S \text{ properly}\}$ . Let  $\pi : \tilde{X} \rightarrow X$  be a projective resolution of singularities. For a curve  $C$  we denote the normalization of  $C$  by  $\nu_C : \tilde{C} \rightarrow C$ . The functor of *formal divisors* on  $C$  is the formal group given by  $\underline{\text{FDiv}}_C = \bigoplus_{p \in C(k)} \underline{\text{Hom}}_{A_b/k}(\hat{\mathcal{O}}_{C,p}^*, k^*)$  (see [Ru3, Def. 2.1]). A finite

morphism  $\zeta : Z \rightarrow C$  induces an obvious *push-forward* of formal divisors  $\zeta_* : \mathbf{FDiv}_Z \rightarrow \mathbf{FDiv}_C$  (see [Ru3, Def. 2.4]). If  $C$  is normal, there is a canonical homomorphism  $\text{fml} : \widehat{\mathbf{Div}}_C \rightarrow \mathbf{FDiv}_C$  given by  $\mathcal{D} \mapsto \sum_{p \in C(k)} (\mathcal{D}, ?)_p$ , where  $(?, ?)_p$  is the local symbol at  $p \in C$  (see [Ru3, Prop. 2.5]).

We use the definition of  $\mathbf{Div}_{\tilde{X}/X}^0$  given in [Ru3, Def. 2.6, 2.7], which coincides with the one given in [Ru1, Prop. 3.23, 3.24], as is easily verified (see [Ru3, Rmk. 2.8]):

**Definition 3.1.** If  $C$  is a projective curve, then

$$\mathbf{Div}_{\tilde{C}/C}^0 = \ker \left( \mathbf{Div}_{\tilde{C}}^0 \xrightarrow{\text{fml}} \mathbf{FDiv}_{\tilde{C}} \xrightarrow{\nu_*} \mathbf{FDiv}_C \right)$$

where  $\nu : \tilde{C} \rightarrow C$  is the normalization. For higher dimensional  $X$

$$\mathbf{Div}_{\tilde{X}/X}^0 = \bigcap_C \left( ? \cdot \tilde{C} \right)^{-1} \mathbf{Div}_{\tilde{C}/C}^0$$

where the intersection ranges over all Cartier curves in  $X$  relative to the singular locus of  $X$  (see [Ru1, Definition 3.1]), and  $(? \cdot \tilde{C})$  is the pull-back of relative Cartier divisors on  $\tilde{X}$  to those on  $\tilde{C}$ .

The functor  $\mathbf{Div}_{\tilde{X}/X}^0$  is a torsion-free dual-algebraic ([Ru2, Def. 1.20]) formal group (see [Ru3, Thm. 4.5] or [Ru1, Prop. 3.24]), this means that the Cartier dual of  $\mathbf{Div}_{\tilde{X}/X}^0$  is a connected algebraic affine group (by definition and [Ln, (5.2)]). Equivalently,  $\mathbf{Div}_{\tilde{X}/X}^0 \cong \mathbb{Z}^t \times (\hat{\mathbb{G}}_a)^v$  for some  $t, v \in \mathbb{N}$  (cf. [Ln, (4.2)]).

As  $\text{char}(k) = 0$ , a formal group  $\mathcal{E}$  is completely determined by its  $k$ -valued points  $\mathcal{E}(k)$  and its Lie-algebra  $\text{Lie}(\mathcal{E})$  (see [Ru1, Cor. 1.7]). If  $\mathcal{E}$  is torsion-free and dual-algebraic, then  $\mathcal{E}(k)$  is a free abelian group of finite rank and  $\text{Lie}(\mathcal{E})$  is a finite dimensional  $k$ -vector space. Then the dimension of the Cartier dual  $\mathcal{E}^\vee$  of  $\mathcal{E}$  is  $\dim \mathcal{E}^\vee = \text{rk } \mathcal{E}(k) + \dim \text{Lie}(\mathcal{E})$  (cf. [Ln, (5.2)]).

**Proposition 3.2** (Bertini's Theorem). *Let  $\mathcal{L}$  be a line bundle on  $X$ . Then for almost all  $C \in |\mathcal{L}|$  the inverse image  $C_{\tilde{X}} = C \times_X \tilde{X} = \pi^{-1}C \in |\pi^*\mathcal{L}|$  is smooth.*

**Proof.** [Ha, II, Theorem 8.18] and induction. ■

**Proposition 3.3.** *Let  $B$  be a variety parametrizing Cartier curves in  $X$ ,  $C_0 \in B$  and  $\mathcal{D} \in \mathbf{Div}_{\tilde{X}}^0$  such that  $(\nu_{C_0,*} \circ \text{fml})(\mathcal{D} \cdot \tilde{C}_0) \neq 0$ . Then there exists an open neighbourhood  $U \ni C_0$  in  $B$  such that  $(\nu_{C,*} \circ \text{fml})(\mathcal{D} \cdot \tilde{C}) \neq 0$  for all  $C \in U$ . In other words: The zero locus of the function  $(\nu_{*,*} \circ \text{fml})(\mathcal{D} \cdot ?) : B \rightarrow \bigoplus_{C \in B} \mathbf{FDiv}_C$  is closed.*

**Proof.** Let  $\mathcal{C} \xrightarrow{\beta} B$  be the universal curve over  $B$ , let  $\mathcal{Z} := \mathcal{C} \times_X \tilde{X}$  be the relative curve over  $B$  of preimages of  $\mathcal{C}$  in  $\tilde{X}$ . We may assume that the fibre  $\mathcal{Z}_{\beta(C_0)}$  of  $\mathcal{Z}$  over the point  $\beta(C_0) \in B$  corresponding to  $C_0$  is normal, i.e.  $\tilde{C}_0 = \mathcal{Z}_{\beta(C_0)}$ . Otherwise this can be achieved by blowing up  $\tilde{X}$ .

If  $C$  is a curve,  $S \subset C(k)$  and  $F \in \mathbf{FDiv}_C$ , we denote by  $F_S := \sum_{p \in S} F_p \in \underline{\text{Hom}}_{\text{Ab}/k}(\hat{\mathcal{O}}_{C,S}^*, k^*)$  the sum of those components  $F_p \in \underline{\text{Hom}}_{\text{Ab}/k}(\hat{\mathcal{O}}_{C,p}^*, k^*)$  with  $p \in S$ .

By definition,  $(\nu_{C_0,*} \circ \text{fml})(\mathcal{D} \cdot \widetilde{C_0}) \neq 0$  implies that there is  $p \in C_0(k)$  and  $f_0 \in \mathcal{O}_{C_0,p}^*$  such that  $\sum_{q \rightarrow p} (\mathcal{D} \cdot \widetilde{C_0}, f_0)_q \neq 0$ , where  $q \rightarrow p$  are the points  $q \in \widetilde{C_0}$  over  $p \in C_0$ . Let  $S(\mathcal{D})$  denote the support of  $\mathcal{D}$ . Then necessarily  $q \in S(\mathcal{D})$ . Let  $\mathbb{L}_R := \mathbb{G}_m(? \otimes R)$  be the Weil restriction of  $\mathbb{G}_{m,R}$  from  $R$  to  $k$ . We are going to construct a regular map  $\Phi : T \rightarrow \mathbb{L}_R$  from a neighbourhood  $T$  of  $\beta(C_0)$  in  $B$  to  $\mathbb{L}_R$ , such that  $\Phi(b) = ((\nu_{C_b,*} \circ \text{fml})(\mathcal{D} \cdot \mathcal{Z}_b))_{\pi S(\mathcal{D}) \cap \mathcal{C}_b} (f|_{\mathcal{C}_b})$  for all  $b \in T$  for some  $f \in \mathcal{O}_{\mathcal{C}_T, \pi S(\mathcal{D})_c}^*$ , and  $\Phi(\beta(C_0)) = ((\nu_{C_0,*} \circ \text{fml})(\mathcal{D} \cdot \widetilde{C_0}))_p (f_0)$ . Then  $\Phi(\beta(C_0)) \neq 0$  implies that there is an open neighbourhood  $U \ni \beta(C_0)$  in  $B$  such that  $\Phi(u) \neq 0$  for all  $u \in U$ , proving the assertion.

Let  $G(\mathcal{D}) \in \text{Ext}_{\text{Ab}/k}(\text{Alb}(\widetilde{X}), \mathbb{L}_R) \cong \text{Pic}_{\text{Alb}(\widetilde{X})}^0(R) \cong \text{Pic}_{\widetilde{X}}^0(R)$  be the algebraic group corresponding to  $\mathcal{O}_{\widetilde{X} \otimes R}(\mathcal{D})$ . The canonical 1-section of  $\mathcal{O}_{\widetilde{X} \otimes R}(\mathcal{D})$  induces a rational map  $\varphi^{\mathcal{D}} : \widetilde{X} \dashrightarrow G(\mathcal{D})$ , which is regular away from  $S(\mathcal{D})$ . If  $f \in \mathcal{O}_{\mathcal{Z}_b,q}^*$  for some  $b \in B$  and  $q \in \mathcal{Z}_b$ , then according to [Ru1, Lem. 3.16]  $(\varphi^{\mathcal{D}}|_{\mathcal{Z}_b}, f)_q$  is contained in the fibre of  $G(\mathcal{D})$  over  $0 \in \text{Alb}(\widetilde{X})$ , which is  $\mathbb{L}_R$ , and  $(\varphi^{\mathcal{D}}|_{\mathcal{Z}_b}, f)_q = (\mathcal{D} \cdot \mathcal{Z}_b, f)_q$ .

Let  $M$  be a modulus for the rational map  $\varphi^{\mathcal{D}}|_{\widetilde{C_0}}$ . By the Approximation Lemma we find  $f_p \in \mathcal{O}_{C_0,p}^*$  with  $f_0/f_p \equiv 1 \pmod{M}$  at all points  $q \rightarrow p$  and  $f_p \equiv 1 \pmod{M}$  at all points of  $S(\mathcal{D}) \cap \widetilde{C_0} \setminus \nu_{C_0}^{-1}(p)$ . Then

$$\sum_{q \rightarrow p} (\varphi^{\mathcal{D}}|_{\widetilde{C_0}}, f_0)_q = \sum_{q \in S(\mathcal{D}) \cap \widetilde{C_0}} (\varphi^{\mathcal{D}}|_{\widetilde{C_0}}, f_p)_q = - \sum_{c \in \widetilde{C_0} \setminus S(\mathcal{D})} v_c(f_p) \varphi^{\mathcal{D}}(c)$$

Let  $T \subset B$  be an affine neighbourhood of  $\beta(C_0)$ , and let  $f \in \mathcal{O}_{\mathcal{C}_T, \pi S(\mathcal{D})_c}^*$  be a lift of  $f_p \in \mathcal{O}_{C_0,p}^*$ . We consider  $f$  as an element of  $\mathcal{O}_{\mathcal{Z}_T, S(\mathcal{D})_{\mathcal{Z}}}^*$ . Let  $S(f)$  denote the support of  $\text{div}(f)$  in  $\mathcal{Z}_T$ . Shrinking  $T$  if necessary, we may assume that  $S(f) \cap S(\mathcal{D})_{\mathcal{Z}} = \emptyset$ , that  $S(f)$  intersects the fibres  $\mathcal{Z}_b$  over  $T$  transversally and that  $S(f) \rightarrow T$  is étale. We choose  $f$  in such a way that  $\beta(C_0) \in T$ . Furthermore we may assume that all fibres  $\mathcal{Z}_b$  over  $T$  are normal by Bertini's Theorem (cf. Proposition 3.2). We obtain a regular map  $\lambda : S(f) \rightarrow G(\mathcal{D})$  defined by

$$\lambda(s) = -v_s(f|_{\mathcal{Z}_{\beta(s)}}) \varphi^{\mathcal{D}}(s) = -v_{E(s)}(f) \varphi^{\mathcal{D}}(s)$$

for  $s \in S(f)$ , where  $E(s)$  is the unique irreducible component of  $S(f)$  containing  $s$ . While  $S(f) \rightarrow T$  is finite étale, taking the trace of  $\lambda$  over  $T$  yields the map  $\text{Tr } \lambda : T \rightarrow \mathbb{L}_R$ , given by

$$\begin{aligned} \text{Tr } \lambda(b) &= \sum_{s \rightarrow b} \lambda(s) = - \sum_{c \in \mathcal{Z}_b \setminus S(\mathcal{D})} v_c(f|_{\mathcal{Z}_b}) \varphi^{\mathcal{D}}(c) = - \sum_{c \in \mathcal{Z}_b \setminus S(\mathcal{D})} (\varphi^{\mathcal{D}}|_{\mathcal{Z}_b}, f|_{\mathcal{Z}_b})_c \\ &= \sum_{q \in \mathcal{Z}_b \cap S(\mathcal{D})} (\varphi^{\mathcal{D}}|_{\mathcal{Z}_b}, f|_{\mathcal{Z}_b})_q = ((\nu_{C_b,*} \circ \text{fml})(\mathcal{D} \cdot \mathcal{Z}_b))_{\pi S(\mathcal{D}) \cap \mathcal{C}_b} (f|_{\mathcal{Z}_b}) \end{aligned}$$

Then  $\text{Tr } \lambda(\beta(C_0)) = ((\nu_{C_0,*} \circ \text{fml})(\mathcal{D} \cdot \widetilde{C_0}))_p (f_0)$ . As  $\lambda$  is regular on  $S(f)$ ,  $\text{Tr } \lambda$  is regular on  $T$ , according to [Se, III, No. 5, Prop. 8]. Thus  $\Phi := \text{Tr } \lambda$  gives the desired map. ■

**Corollary 3.4.** *Let  $\mathcal{E} \cong \mathbb{Z}^t \times_k (\widehat{\mathbb{G}}_a)^v$  be a formal subgroup of  $\text{Div}_{\widetilde{X}}^0$  which contains  $\text{Div}_{\widetilde{X}/X}^0$ , and let  $B$  be a variety parametrizing Cartier curves in  $X$ . The function  $\dim(\mathcal{E} \times_{\text{Div}_{\widetilde{X}}} \text{Div}_{\widetilde{X}/X}^0)^\vee : B \rightarrow \mathbb{N}$  is upper semi-continuous.*



**Proof.** For  $C \in B$  write  $d(C) := \dim(\mathcal{E} \times_{\underline{\text{Div}}_{\widetilde{C}}} \underline{\text{Div}}_{\widetilde{C}/C}^0)^\vee$ . We have to show that the sets  $\{C \in B \mid d(C) \leq n\}$  are open for all  $n \in \mathbb{N}$ . Let  $C_0 \in B$  with  $d(C_0) = n$ . We show: there is an open neighbourhood  $U \ni C_0$  in  $B$  such that  $d(C) \leq n$  for all  $C \in U$ .

For some  $c_0, c_1 \in \mathbb{N}$  with  $c_0 + c_1 = c := \dim \mathcal{E}^\vee - n$  one finds  $\mathbb{Z}$ -linearly independent elements  $\delta_1^{(0)}, \dots, \delta_{c_0}^{(0)} \in (\mathcal{E} \cdot \widetilde{C}_0)(k) \setminus \underline{\text{Div}}_{\widetilde{C}_0/C_0}^0(k)$  and  $k$ -linearly independent elements  $\delta_1^{(1)}, \dots, \delta_{c_1}^{(1)} \in \text{Lie}(\mathcal{E} \cdot \widetilde{C}_0) \setminus \text{Lie}(\underline{\text{Div}}_{\widetilde{C}_0/C_0}^0)$  that extend a basis of  $\underline{\text{Div}}_{\widetilde{C}_0/C_0}^0(k)$  resp.  $\text{Lie}(\underline{\text{Div}}_{\widetilde{C}_0/C_0}^0)$  to a basis of  $\mathcal{E}(k)$  resp.  $\text{Lie}(\mathcal{E})$ . Let  $\mathcal{D}_j^{(i)} \in \mathcal{E}$  with  $\mathcal{D}_j^{(i)} \cdot \widetilde{C}_0 = \delta_j^{(i)}$  for  $j = 1, \dots, c_i$  and  $i = 0, 1$ . By Proposition 3.3 there exists an open neighbourhood  $U \ni C_0$  in  $T$  such that we have  $(\nu_{C,*} \circ \text{fml})(\mathcal{D}_j^{(i)} \cdot \widetilde{C}) \neq 0$  for  $j = 1, \dots, c_i$  and  $i = 0, 1$  for all  $C \in U$ , and the locus in  $B$  where  $(\nu_{C,*} \circ \text{fml})(\mathcal{D}_1^{(i)} \cdot \widetilde{C}), \dots, (\nu_{C,*} \circ \text{fml})(\mathcal{D}_{c_i}^{(i)} \cdot \widetilde{C})$  are linearly dependent mod  $\underline{\text{Div}}_{\widetilde{C}/C}^0$  is a closed proper subset  $V \subset U$ . Then  $d(C) \leq \dim \mathcal{E}^\vee - c = n$  for all  $C \in U \setminus V$ . ■

**Proposition 3.5.** *Let  $\mathcal{L}$  be an ample line bundle on  $X$ . Then there exists  $N \in \mathbb{N}$  such that  $\underline{\text{Div}}_{\widetilde{X}/X}^0$  can be computed from curves in the parameter space  $B := |\mathcal{L}^N|_S^{d-1}$ : this means by definition*

$$\underline{\text{Div}}_{\widetilde{X}/X}^0 = \bigcap_{C \in B} \left( ? \cdot \widetilde{C} \right)^{-1} \underline{\text{Div}}_{\widetilde{C}/C}^0.$$

**Proof.** Cartier curves in  $X$  are complete intersections locally in a neighbourhood of the singular locus  $S$ . For the computation of  $\underline{\text{Div}}_{\widetilde{X}/X}^0$  only the formal neighbourhood of  $S$  is relevant, since  $\underline{\text{Div}}_{\widetilde{X}/X}^0$  is defined via the push-forward of formal divisors (see Definition 3.1) and its support is contained in  $S_{\widetilde{X}} := S \times_X \widetilde{X}$ . Thus we can replace the range of all Cartier curves by a set of complete intersection curves. As  $\underline{\text{Div}}_{\widetilde{X}/X}^0$  is dual-algebraic, there exists an effective divisor  $E$  on  $\widetilde{X}$  with support in  $S_{\widetilde{X}}$  such that  $\underline{\text{Div}}_{\widetilde{X}/X}^0 \subset \mathcal{F}_{\widetilde{X},E}$  (see [Ru2, Prop. 3.21] or proof of [Ru1, Prop. 3.24]), where  $\mathcal{F}_{\widetilde{X},E}$  is the formal group associated with the modulus  $E$  (see [Ru2, Def. 3.13]). If  $m$  is the maximum multiplicity of  $E$ , then only the  $(m-1)^{\text{th}}$ -infinitesimal neighbourhood of  $S$  is relevant. (This follows e.g. from [Ru1, Lem. 3.21].) Any Cartier curve  $C \subset X$  can be approximated by a curve  $C_N \in |\mathcal{L}^N|$  such that the  $(m-1)^{\text{th}}$ -infinitesimal neighbourhood of  $S \cdot C$  coincides with the one of  $S \cdot C_N$  for sufficiently large  $N \in \mathbb{N}$ . ■

**Proposition 3.6.** *Let  $\mathcal{E} \cong \mathbb{Z}^t \times_k (\widehat{\mathbb{G}}_a)^v$  be a formal subgroup of  $\underline{\text{Div}}_{\widetilde{X}}^0$  containing  $\underline{\text{Div}}_{\widetilde{X}/X}^0$ , and let  $B$  a parameter space of curves such that  $\underline{\text{Div}}_{\widetilde{X}/X}^0$  can be computed from  $B$ . Then there are finitely many curves  $C_1, \dots, C_r \in B$  such that*

$$\underline{\text{Div}}_{\widetilde{X}/X}^0 = \bigcap_{i=1}^r \left( ? \cdot \widetilde{C}_i \right) \Big|_{\mathcal{E}}^{-1} \underline{\text{Div}}_{\widetilde{C}_i/C_i}^0.$$

**Proof.** For  $C \in B$  set  $\mathcal{F}_C := \left( ? \cdot \widetilde{C} \right) \Big|_{\mathcal{E}}^{-1} \underline{\text{Div}}_{\widetilde{C}/C}^0$ . Then it holds  $\underline{\text{Div}}_{\widetilde{X}/X}^0 = \bigcap_{C \in B} \mathcal{F}_C$ . For every sequence  $\{C_\nu\}$  of curves in  $B$  the sequence  $\{\mathcal{E}_\nu\}$  with

$\mathcal{E}_0 := \mathcal{E}$ ,  $\mathcal{E}_{\nu+1} := \mathcal{E}_\nu \cap \mathcal{F}_{C_\nu}$  becomes stationary, due to the noetherian properties of the formal group  $\mathcal{E}$ , cf. [Ru1, Remark 3.26]. ■

**Corollary 3.7.** *If  $C_i = H_1^{(i)} \cap \dots \cap H_{d-1}^{(i)}$ , then  $C_0 := (\sum_i H_1^{(i)}) \cap \dots \cap (\sum_i H_{d-1}^{(i)})$  satisfies:*

$$\underline{\mathrm{Div}}_{\widetilde{X}/X}^0 = \left( ? \cdot \widetilde{C}_0 \right) \Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}_{\widetilde{C}_0/C_0}^0.$$

**Theorem 3.8.** *Let  $\mathcal{E} \cong \mathbb{Z}^t \times_k (\widehat{\mathbb{G}}_a)^v$  be a formal subgroup of  $\underline{\mathrm{Div}}_{\widetilde{X}}^0$  such that  $\underline{\mathrm{Div}}_{\widetilde{X}/X}^0 \subset \mathcal{E}$ . Let  $\mathcal{L}$  be an ample line bundle on  $X$ . For  $C \in |\mathcal{L}^N|_S^{d-1}$  and sufficiently large  $N$  the property*

$$\underline{\mathrm{Div}}_{\widetilde{X}/X}^0 = \left( ? \cdot \widetilde{C} \right) \Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}_{\widetilde{C}/C}^0$$

*is open and dense.*

**Proof.** By Propositions 3.5, 3.6 and Corollary 3.7 there exist numbers  $r, N \in \mathbb{N}$  and a curve  $C_0 \in |\mathcal{L}^{rN}|_S^{d-1}$  with  $\underline{\mathrm{Div}}_{\widetilde{X}/X}^0 = \left( ? \cdot \widetilde{C}_0 \right) \Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}_{\widetilde{C}_0/C_0}^0$ . Let  $C$  be a general curve in  $|\mathcal{L}^{rN}|_S^{d-1}$ . By definition of  $\underline{\mathrm{Div}}_{\widetilde{X}/X}^0$  we have

$$\left( ? \cdot \widetilde{C} \right) \Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}_{\widetilde{C}/C}^0 \supset \underline{\mathrm{Div}}_{\widetilde{X}/X}^0 = \left( ? \cdot \widetilde{C}_0 \right) \Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}_{\widetilde{C}_0/C_0}^0,$$

i.e.  $\dim \left( \mathcal{E} \times_{\underline{\mathrm{Div}}_{\widetilde{C}}} \underline{\mathrm{Div}}_{\widetilde{C}/C}^0 \right)^\vee \geq \dim \left( \mathcal{E} \times_{\underline{\mathrm{Div}}_{\widetilde{C}_0}} \underline{\mathrm{Div}}_{\widetilde{C}_0/C_0}^0 \right)^\vee$ . According to Proposition 3.4 the expression  $\dim \left( \mathcal{E} \times_{\underline{\mathrm{Div}}_{\widetilde{C}}} \underline{\mathrm{Div}}_{\widetilde{C}/C}^0 \right)^\vee$ , as a function in  $C \in B$ , is upper semi-continuous. Thus a general curve  $C \in B$  satisfies

$$\left( ? \cdot \widetilde{C} \right) \Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}_{\widetilde{C}/C}^0 = \left( ? \cdot \widetilde{C}_0 \right) \Big|_{\mathcal{E}}^{-1} \underline{\mathrm{Div}}_{\widetilde{C}_0/C_0}^0 = \underline{\mathrm{Div}}_{\widetilde{X}/X}^0.$$

■

## 4 Example: Product of two Cuspidal Curves

We conclude this paper with the discussion of an example that was the subject of the diploma of Alexander Schwarzhaupt [Sch]. This example illustrates some pathological properties: the Albanese of Esnault-Srinivas-Viehweg is not in general compatible with products, in this example we obtain (writing  $\mathrm{Alb}^{\mathrm{ESV}}(X) := \mathrm{Alb}(X, X_{\mathrm{sing}})$ )

$$\dim \left( \mathrm{Alb}^{\mathrm{ESV}}(\Gamma_\alpha \times \Gamma_\beta) \right) > \dim \left( \mathrm{Alb}^{\mathrm{ESV}}(\Gamma_\alpha) \times \mathrm{Alb}^{\mathrm{ESV}}(\Gamma_\beta) \right).$$

Moreover, given a very ample line bundle  $\mathcal{L}$  on the surface  $X = \Gamma_\alpha \times \Gamma_\beta$  and a curve  $C_N \in |\mathcal{L}^N|$  in general position, we work out a necessary and sufficient condition on the integer  $N$  for the surjectivity of the Gysin map  $\mathrm{Alb}^{\mathrm{ESV}}(C_N) \longrightarrow \mathrm{Alb}^{\mathrm{ESV}}(X)$ .

The base field  $k$  is assumed to be algebraically closed and of characteristic 0.

## 4.1 Cuspidal Curve

Let  $\Gamma_\alpha \subset \mathbb{P}_k^2$  be the projective curve defined by

$$\Gamma_\alpha : X^{2\alpha+1} - Y^2 Z^{2\alpha-1} = 0$$

where  $X : Y : Z$  are homogeneous coordinates of  $\mathbb{P}_k^2$  and  $\alpha \geq 1$  is an integer. The singularities of this curve are cusps at  $0 := [0 : 0 : 1]$  and  $\infty := [0 : 1 : 0]$ . The normalization  $\tilde{\Gamma}_\alpha$  of  $\Gamma_\alpha$  is the projective line:  $\tilde{\Gamma}_\alpha = \mathbb{P}_k^1$ . Then  $\text{Alb}(\tilde{\Gamma}_\alpha) = \text{Alb}(\mathbb{P}_k^1) = 0$ . Since  $\text{Alb}^{\text{ESV}}(\Gamma_\alpha)$  is an extension of  $\text{Alb}(\tilde{\Gamma}_\alpha)$  by the linear group  $L_{\Gamma_\alpha} = (\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0)^\vee$ , we obtain

$$\text{Alb}^{\text{ESV}}(\Gamma_\alpha) = L_{\Gamma_\alpha} = \left( \underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0 \right)^\vee.$$

Moreover,  $\Gamma_\alpha$  is homeomorphic to  $\mathbb{P}_k^1$ , i.e. the normalization  $\tilde{\Gamma}_\alpha$  is given by the largest homeomorphic curve  $\Gamma'_\alpha$ . This implies that  $L_{\Gamma_\alpha}$  is a unipotent group (see Theorem 1.5) and equivalently  $\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0$  is an infinitesimal formal group (see [Ru2, Proposition 1.17]). We have  $\text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0) = \text{Hom}_k(L_{\Gamma_\alpha}(k), k)$ .

The  $k$ -valued points of  $L_{\Gamma_\alpha}$  are given by (see Theorem 1.5)

$$L_{\Gamma_\alpha}(k) = \frac{1 + \mathfrak{m}_{\tilde{\Gamma}_\alpha,0}}{1 + \mathfrak{m}_{\Gamma_\alpha,0}} \times \frac{1 + \mathfrak{m}_{\tilde{\Gamma}_\alpha,\infty}}{1 + \mathfrak{m}_{\Gamma_\alpha,\infty}} \cong \frac{\mathfrak{m}_{\tilde{\Gamma}_\alpha,0}}{\mathfrak{m}_{\Gamma_\alpha,0}} \oplus \frac{\mathfrak{m}_{\tilde{\Gamma}_\alpha,\infty}}{\mathfrak{m}_{\Gamma_\alpha,\infty}}.$$

The dimensions are computed in [Sch, Proposition 1.5] as

$$\begin{aligned} \dim_k(\mathfrak{m}_{\tilde{\Gamma}_\alpha,0}/\mathfrak{m}_{\Gamma_\alpha,0}) &= \alpha \\ \dim_k(\mathfrak{m}_{\tilde{\Gamma}_\alpha,\infty}/\mathfrak{m}_{\Gamma_\alpha,\infty}) &= 2\alpha(\alpha - 1) \end{aligned}$$

hence

**Proposition 4.1.**

$$\dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha) = \dim L_{\Gamma_\alpha} = \dim_k \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0) = \alpha(2\alpha - 1).$$

As  $\text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0) = \text{fml}^{-1} \bigoplus_{q=0,\infty} \text{Hom}_k(\mathfrak{m}_{\tilde{\Gamma}_\alpha,q}/\mathfrak{m}_{\Gamma_\alpha,q}, k)$  it holds

$$\# \text{v}_q(\text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0)) = \#(\text{v}_q(\mathcal{O}_{\tilde{\Gamma}_\alpha}) \setminus \text{v}_q(\mathcal{O}_{\Gamma_\alpha})) = \dim_k(\mathfrak{m}_{\tilde{\Gamma}_\alpha,q}/\mathfrak{m}_{\Gamma_\alpha,q}),$$

for  $q \in \{0, \infty\} \subset \tilde{\Gamma}_\alpha$ , cf. [Ru1, No. 3.3]. A basis of  $\text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0)$  is given by the following set of representatives

$$\Theta_{\Gamma_\alpha} = \left\{ t_q^{-\nu} \mid -\nu \in \text{v}_q(\text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0)), q = 0, \infty \right\}$$

where  $t_q$  is a local parameter of  $\mathfrak{m}_{\tilde{\Gamma}_\alpha,q}$  at  $q$  and  $t_q^{-\nu} \in \mathcal{O}_{\tilde{\Gamma}_\alpha,p}$  for all  $p \neq q$ .

## 4.2 Cuspidal Surface

Let  $X$  be the product of the cuspidal curves  $\Gamma_\alpha, \Gamma_\beta$  from Subsection 4.1:

$$X = \Gamma_\alpha \times \Gamma_\beta$$

where  $\alpha, \beta \geq 1$  are integers. The singular locus of  $X$  is

$$X_{\text{sing}} = (\{0\} \times \Gamma_\beta) \cup (\{\infty\} \times \Gamma_\beta) \cup (\Gamma_\alpha \times \{0\}) \cup (\Gamma_\alpha \times \{\infty\}).$$

The normalization  $\tilde{X}$  of  $X$  is a resolution of singularities and given by  $\tilde{X} = \tilde{\Gamma}_\alpha \times \tilde{\Gamma}_\beta = \mathbb{P}_k^1 \times \mathbb{P}_k^1$ . Then  $\text{Alb}(\tilde{X}) = \text{Alb}(\mathbb{P}_k^1) \times \text{Alb}(\mathbb{P}_k^1) = 0$ . Thus the Albanese of Esnault-Srinivas-Viehweg of  $X$  coincides with its affine part:

$$\text{Alb}^{\text{ESV}}(X) = L_X = \left( \underline{\text{Div}}_{\tilde{X}/X}^0 \right)^\vee$$

and  $\underline{\text{Div}}_{\tilde{X}/X}^0$  is an infinitesimal formal group, since the normalization is a homeomorphism. The task is now to determine  $\underline{\text{Div}}_{\tilde{X}/X}$ . The support of  $\underline{\text{Div}}_{\tilde{X}/X}^0$  is the preimage of  $X_{\text{sing}}$ :

$$\text{Supp} \left( \underline{\text{Div}}_{\tilde{X}/X}^0 \right) = (\{0\} \times \tilde{\Gamma}_\beta) \cup (\{\infty\} \times \tilde{\Gamma}_\beta) \cup (\tilde{\Gamma}_\alpha \times \{0\}) \cup (\tilde{\Gamma}_\alpha \times \{\infty\}).$$

By Proposition 3.8 we can compute  $\underline{\text{Div}}_{\tilde{X}/X}^0$  by the preimage of  $\underline{\text{Div}}_{\tilde{C}/C}^0$  in  $\underline{\text{Div}}_{\tilde{X}}$  under pull-back to  $\tilde{C}$ , for a sufficiently ample curve  $C \subset X$  in general position. We may take  $C = \bigcup_{i=1}^k (\{p_i\} \times \Gamma_\beta) \cup \bigcup_{j=1}^l (\Gamma_\alpha \times \{q_j\})$  for sufficiently many points  $p_i \in \Gamma_\alpha \setminus \{0, \infty\}$  and  $q_j \in \Gamma_\beta \setminus \{0, \infty\}$ . Then  $\underline{\text{Div}}_{\tilde{C}/C}^0 = \prod_{i=1}^k \underline{\text{Div}}_{\tilde{\Gamma}_\beta/\Gamma_\beta}^0 \times \prod_{j=1}^l \underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0$ . Now  $\text{Lie}(\underline{\text{Div}}_{\tilde{X}/X}^0)$  is the space of those  $\delta \in \text{Lie}(\underline{\text{Div}}_{\tilde{X}}^0)$  with support in the preimage of  $X_{\text{sing}}$  such that  $\delta \cdot (\{p\} \times \tilde{\Gamma}_\beta) \in \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\beta/\Gamma_\beta}^0)$  and  $\delta \cdot (\tilde{\Gamma}_\alpha \times \{q\}) \in \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0)$  for all  $p \in \tilde{\Gamma}_\alpha \setminus \{0, \infty\}$  and all  $q \in \tilde{\Gamma}_\beta \setminus \{0, \infty\}$ . From this we see: If  $\Theta_{\Gamma_\iota}$  is a set of representatives of a basis of  $\text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\iota/\Gamma_\iota}^0)$  for  $\iota = \alpha, \beta$ , then

$$\Theta_{\Gamma_\alpha \times \Gamma_\beta} = \left\{ \vartheta_{\Gamma_\alpha} \otimes \vartheta_{\Gamma_\beta} \mid (\vartheta_{\Gamma_\alpha}, \vartheta_{\Gamma_\beta}) \in \left( (\Theta_{\Gamma_\alpha} \cup \{1\}) \times (\Theta_{\Gamma_\beta} \cup \{1\}) \right) \setminus \{(1, 1)\} \right\}$$

is a set of representatives of a basis of  $\text{Lie}(\underline{\text{Div}}_{\tilde{X}/X}^0)$ .

Thus the dimension of  $\text{Alb}^{\text{ESV}}(X)$  is given by

$$\begin{aligned} \dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) &= \dim_k \text{Lie} \left( \underline{\text{Div}}_{(\tilde{\Gamma}_\alpha \times \tilde{\Gamma}_\beta)/(\Gamma_\alpha \times \Gamma_\beta)}^0 \right) \\ &= \left( \dim_k \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0) + 1 \right) \cdot \left( \dim_k \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\beta/\Gamma_\beta}^0) + 1 \right) - 1 \\ &= (\alpha(2\alpha - 1) + 1) \cdot (\beta(2\beta - 1) + 1) - 1. \end{aligned}$$

With Proposition 4.1 this yields

**Proposition 4.2.**

$$\dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta) = (\dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha) + 1) \cdot (\dim \text{Alb}^{\text{ESV}}(\Gamma_\beta) + 1) - 1.$$

We obtain a basis of  $\text{im} \left( \text{Lie}(\underline{\text{Div}}_{\tilde{X}/X}^0) \rightarrow (\mathcal{K}_{\tilde{X}}/\mathcal{O}_{\tilde{X}})_{\tilde{\Gamma}_\alpha \times \{q\}} \right)$  for  $q \in \{0, \infty\} \subset \tilde{\Gamma}_\beta$  from the following set of representatives

$$\Theta_{\Gamma_\alpha \times q} = \left\{ \vartheta_{\Gamma_\alpha} \otimes t_q^{-\nu} \mid \vartheta_{\Gamma_\alpha} \in (\Theta_{\Gamma_\alpha} \cup \{1\}), -\nu \in \mathfrak{v}_q \left( \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\beta/\Gamma_\beta}^0) \right) \right\}.$$

Now  $\mathfrak{v}_q \left( \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\beta/\Gamma_\beta}^0) \right) = \mathfrak{v}_{\tilde{\Gamma}_\alpha \times \{q\}} \left( \text{Lie}(\underline{\text{Div}}_{\tilde{X}/X}^0) \right)$ , therefore

**Proposition 4.3.**

$$\dim_k \left( \text{Lie} \left( \underline{\text{Div}}_{\tilde{X}/X}^0 \right) \right)_{\tilde{\Gamma}_\alpha \times \{q\}}^{-\nu} = \dim_k \text{Lie} \left( \underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0 \right) + 1$$

for  $q \in \{0, \infty\} \subset \tilde{\Gamma}_\beta$  and all  $-\nu \in \mathfrak{v}_{\tilde{\Gamma}_\alpha \times \{q\}} \left( \text{Lie} \left( \underline{\text{Div}}_{\tilde{X}/X}^0 \right) \right)$ , where we use the notation from Lemma 2.2. Analogously for  $\{p\} \times \tilde{\Gamma}_\beta$ ,  $p \in \{0, \infty\} \subset \tilde{\Gamma}_\alpha$ .

### 4.3 Gysin Map

Consider the following divisor  $D^{k,l}$  on  $X$

$$D^{k,l} = \sum_{i=1}^k (\{p_i\} \times \Gamma_\beta) + \sum_{j=1}^l (\Gamma_\alpha \times \{q_j\})$$

where  $p_i \in \Gamma_\alpha \setminus \{0, \infty\}$  for  $i = 1, \dots, k$  and  $q_j \in \Gamma_\beta \setminus \{0, \infty\}$  for  $j = 1, \dots, l$ . The normalization of  $D^{k,l}$  is isomorphic to the disjoint union of  $k + l$  copies of  $\mathbb{P}_k^1$ . Therefore the Picard variety of the normalization  $\text{Pic}^0 \widetilde{D^{k,l}}$  is trivial. Then by Theorem 1.6, using the explicit formulas of Propositions 1.4 and 1.5,

$$\text{Pic}^0 D^{k,l} = T \times V$$

where  $T \cong (\mathbb{G}_m)^t$  is a torus of rank

$$t = \#S_2 - \#\text{Cp}(D^{k,l}) + 1 = k \cdot l - (k + l) + 1 = (k - 1) \cdot (l - 1),$$

and  $V \cong (\mathbb{G}_a)^v$  is a vectorial group of dimension

$$v = k \cdot \dim L_{\Gamma_\beta} + l \cdot \dim L_{\Gamma_\alpha} = k \cdot \beta(2\beta - 1) + l \cdot \alpha(2\alpha - 1).$$

For general  $p_i \in \Gamma_\alpha$  and  $q_j \in \Gamma_\beta$  the divisor  $D^{2\alpha+1, 2\beta+1}$  is very ample (see [Sch] Lemma 3.2). Set  $\mathcal{L} = \mathcal{O}(D^{2\alpha+1, 2\beta+1})$  and choose  $C_N \in |\mathcal{L}^N|$  in general position. As  $\dim \text{Pic}^0 C_N = \text{const.}$  and  $\dim \text{Pic}^0(C_N \times_X \tilde{X}) = \text{const.}$  among  $C_N \in |\mathcal{L}^N|$  and  $C_N \times_X \tilde{X} = C_{N'}$  is the semi-normalization, the dimension of the vectorial part  $V_{C_N} = \ker(\text{Pic}^0 C_N \rightarrow \text{Pic}^0 C_{N'})$  of  $\text{Pic}^0 C_N = \text{Alb}^{\text{ESV}}(C_N)$  is constant among  $C_N \in |\mathcal{L}^N|$  and hence

$$\dim V_{C_N} = \dim V_{N, D^{2\alpha+1, 2\beta+1}} = N(\alpha(2\alpha - 1)(2\beta + 1) + \beta(2\beta - 1)(2\alpha + 1)).$$

Since  $\text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta)$  is a vectorial group, the map  $\text{Alb}^{\text{ESV}}(C_N) \rightarrow \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta)$  cannot be surjective if  $\dim V_{C_N} < \dim \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta)$ .

Therefore a comparison of dimensions yields:

**Proposition 4.4.** *The Gysin map  $\text{Alb}^{\text{ESV}}(C_N) \rightarrow \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta)$  is not surjective for*

$$N < \frac{(\alpha(2\alpha - 1) + 1) \cdot (\beta(2\beta - 1) + 1) - 1}{\alpha(2\alpha - 1)(2\beta + 1) + \beta(2\beta - 1)(2\alpha + 1)}$$

In the case  $\alpha = \beta$ , this expression simplifies to

$$N < \frac{\alpha(2\alpha - 1) + 2}{2(2\alpha + 1)}.$$

The homomorphism of vectorial groups  $V_{C_N} \rightarrow V_X = \text{Alb}^{\text{ESV}}(X)$  is dual to the map between Lie algebras  $? \cdot \tilde{C}_N : \text{Lie}(\underline{\text{Div}}_{\tilde{X}/X}^0) \rightarrow \text{Lie}(\underline{\text{Div}}_{\tilde{C}_N/C_N}^0)$ , and the surjectivity of the first homomorphism is equivalent to the injectivity of the latter one. Here Definition 3.1 of  $\underline{\text{Div}}_{\tilde{X}/X}^0$  implies immediately that the image of  $\underline{\text{Div}}_{\tilde{X}/X}^0$  under pull-back  $? \cdot \tilde{C} : \underline{\text{Div}}_{\tilde{X}/X}^0 \rightarrow \underline{\text{Div}}_{\tilde{C}}^0$  is contained in  $\underline{\text{Div}}_{\tilde{C}/C}^0$ .

The estimation of Proposition 4.4 yields a necessary condition for surjectivity of the Gysin map, i.e. a bound for  $N$  from below.

The criterion of Lemma 2.2 gives a sufficient condition for the surjectivity of the Gysin map:  $\# \left( \tilde{C}_N \cap (\tilde{\Gamma}_\alpha \times \{q\}) \right) \geq \dim_k \left( \text{Lie}(\underline{\text{Div}}_{\tilde{X}/X}^0) \right)_{\tilde{\Gamma}_\alpha \times \{q\}}^{-\nu}$

for all  $-\nu \in v_{\tilde{\Gamma}_\alpha \times \{q\}}(\text{Lie} \underline{\text{Div}}_{\tilde{X}/X}^0)$ , all  $q \in \{0, \infty\} \subset \tilde{\Gamma}_\beta$ , and the same formula with  $(\tilde{\Gamma}_\alpha \times \{q\})$  replaced by  $(\{p\} \times \tilde{\Gamma}_\beta)$  for all  $p \in \{0, \infty\} \subset \tilde{\Gamma}_\alpha$ . Since  $\dim_k \left( \text{Lie}(\underline{\text{Div}}_{\tilde{X}/X}^0) \right)_{\tilde{\Gamma}_\alpha \times \{q\}}^{-\nu} = \dim_k \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0) + 1$  (see Proposition 4.3) this is equivalent to  $\# \left( \tilde{C}_N \cap (\tilde{\Gamma}_\alpha \times \{q\}) \right) \geq \dim_k \text{Lie}(\underline{\text{Div}}_{\tilde{\Gamma}_\alpha/\Gamma_\alpha}^0) + 1$ .

Then since  $\# \left( \tilde{C}_N \cap (\tilde{\Gamma}_\alpha \times \{q\}) \right) = N \deg(D^{2\alpha+1, 2\beta+1})_{\tilde{X}} = N(2\alpha + 1 + 2\beta + 1) = N2(\alpha + \beta + 1)$ , where  $(D^{k,l})_{\tilde{X}} = \sum_{i=1}^k (\{p_i\} \times \tilde{\Gamma}_\beta) + \sum_{j=1}^l (\tilde{\Gamma}_\alpha \times \{q_j\})$ , we obtain with Proposition 4.1

**Proposition 4.5.** *The Gysin map  $\text{Alb}^{\text{ESV}}(C_N) \rightarrow \text{Alb}^{\text{ESV}}(\Gamma_\alpha \times \Gamma_\beta)$  is surjective if*

$$N \geq \frac{\alpha(2\alpha - 1) + 1}{2(\alpha + \beta + 1)} \quad \text{and} \quad N \geq \frac{\beta(2\beta - 1) + 1}{2(\alpha + \beta + 1)}.$$

For  $\alpha = \beta$  this condition is necessary and sufficient.

**Proof.** The first statement follows from the discussion above. In the case  $\alpha = \beta$  we need to show that the estimation above and the formula from Proposition 4.4 yield the same bound for  $N \in \mathbb{N}$ . As the difference is  $\frac{1}{2(2\alpha+1)} < 1$ , it suffices to check that  $2(2\alpha+1)$  does not divide  $\alpha(2\alpha-1)+1 = \alpha(2\alpha+1) - (2\alpha-1)$ , which is obvious. ■

In [ESV, Variant 6.4] the following sufficient condition for surjectivity of the Gysin map is given:

$$\dim_k \text{im} \left( H^0(X, \mathcal{L}^N) \rightarrow H^0(Z, \mathcal{L}^N|_Z) \right) \geq 2 \dim L_{C_N} + \# \text{Cp}(X) + 2$$

for all  $Z \in \text{Cp}(X)$ , where  $L_{C_N}$  is the largest connected affine subgroup of  $\text{Pic}^0 C_N$  for  $C_N \in |\mathcal{L}^N|$  in general position. For  $X = \Gamma_\alpha \times \Gamma_\beta$  it holds  $\text{Cp}(X) = \{X\}$  and  $\text{Pic}^0 C_N = L_{C_N} = V_{C_N}$ . Alexander Schwarzhaupt showed in his diploma [Sch] that in our example and for  $\alpha = \beta$  this condition leads to the estimation

$$N > 2 \frac{3\alpha(2\alpha - 1) - 1}{2\alpha + 1} + 1.$$

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